Fourier Series Representation of Periodic Signals

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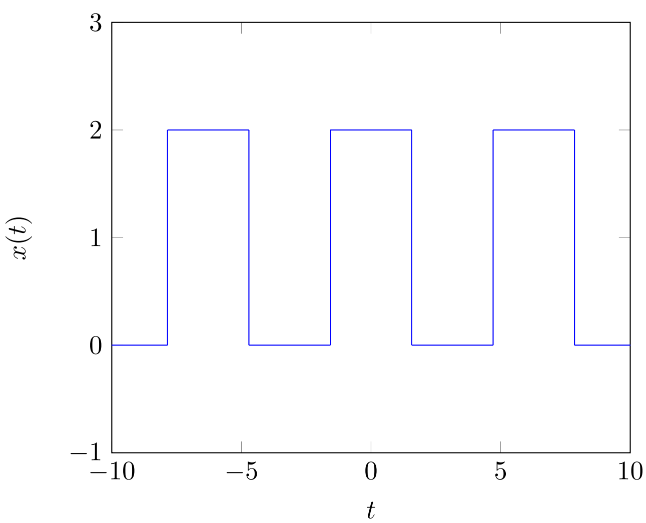
1. What is the Fourier Series for f(t) = 1 + $sin^2(t)$?

$$f(t) = 1 + sin^2(t) = \frac{3}{2} - \frac{1}{2}cos(2t)$$

Since this function is periodic, so it has expression of fourier series. and it has finitely many term(infinite sum is not needed for this case).Thus, Fourier Series for f(t) = 1 + $sin^2(t)$ is $\frac{3}{2} - \frac{1}{2}cos(2t)$

2. Graph the function f(t) which is even, periodic of period 2π and such that f(t) = 2 for 0 < t < π/2 and f(t) = 0 for π/2 < t < π. is the function even, odd, or neither?



Definition of even function
$f(x) = f(-x)$
Geometrically, the graph of an even function is symmetric with respect to the y-axis, meaning that its graph remains unchanged after reflection about the y-axis.

Since for all t f(t) = =f(-t), and it is symmetric with respect to y axis. The function f(t) is even

3. Compute the Fourier Series for the function in Q.2.

f(t) is even function. Since function f(t) is even, $b_n = 0$ for all n > 0. So the nonzero coefficients are the a_n and a_0.

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos{(nx)} + b_n \sin{(nx)}]$$

$$a_0 = \frac{1}{2\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos(nx)dx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 2dt = \frac{1}{2\pi} * 2t\Big|_{-\pi/2}^{\pi/2}$$

$$a_0 = \frac{1}{2\pi} * 2[\pi/2 - (-\pi/2)] = \frac{1}{2\pi} * 2[\pi] = 1$$

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(t)\cos(nt)dt = \frac{2}{\pi} \int_{0}^{\pi/2} 2\cos(nt)dt$$

$$a_n = \frac{4}{\pi} \int_{0}^{\pi/2} \cos(nt)dt = \frac{4}{\pi * n} sin(nt)\Big|_{0}^{\pi/2}$$

If n is even, a_n is zero. if n is odd, then this alternates between + $\frac{4}{n\pi}$ and -$\frac{4}{n\pi}$

Using

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos{(nx)} + b_n \sin{(nx)}]$$

$$f(t) = 1 + \frac{4}{\pi} * cos(t) - \frac{4}{3\pi} * cos(3t) + \frac{4}{5\pi} * cos(5t) - \frac{4}{7\pi} * cos(7t) + ...$$

4. Compute the Fourier Series for the following function

$$f(x) = \begin{cases} 0 & \text{-}\pi \le \text{x} < 0 \\ x & 0 \le \text{x} \le \pi \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos{(nx)} + b_n \sin{(nx)}]$$

$$a_n = \frac{1}{\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos(nx)dx$$

$$b_n = \frac{1}{\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin(nx)dx$$

$$a_0 = \frac{1}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} 0dx + \frac{1}{2\pi}\int_{0}^{\pi} xdx = \frac{1}{2\pi}\int_{0}^{\pi} xdx$$

$$a_0 = \frac{1}{2\pi} * \frac{x^2}{2}\Big|_0^\pi = \frac{1}{4\pi} * [\pi^2 - 0] = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} 0\cos(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)dx$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)dx$$

Integration by parts

$$\int x\cos nx dx = \frac{x\sin nx}{n} - \int \frac{x\sin nx}{n}dx = \frac{x\sin nx}{n} + \frac{\cos nx}{n^2}$$

$$a_n = \frac{1}{\pi} * [\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}]\Big|_0^\pi$$

$$a_n = \frac{1}{\pi} * [\frac{\cos n\pi}{n^2} - \frac{\cos n0}{n^2}] = \frac{1}{n^2\pi} * [\cos n\pi - 1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} 0\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)dx = \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)dx$$

$$b_n = \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)dx$$

Integration by parts

$$\int x\sin nx dx = -\frac{x\cos nx}{n} - \int \frac{x\cos nx}{n}dx = -\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}$$

$$b_n = \frac{1}{\pi} * [-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}]\Big|_0^\pi$$

$$b_n = \frac{1}{\pi} * [-\frac{\pi\cos n\pi}{n}] = -\frac{\cos n\pi}{n}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi} * [\cos n\pi - 1] * cos\,(nx) - \frac{\cos n\pi}{n} * sin\,(nx)\right]$$

$$f(x) = \frac{\pi}{4} - \frac{2cos(x)}{\pi} - \frac{2cos(3x)}{9\pi} - \frac{2cos(5x)}{25\pi} + sin(x) - \frac{sin(2x)}{2} +$$

$$\frac{sin(3x)}{3} - \frac{sin(4x)}{4} + \frac{sin(5x)}{5} + ...$$

5. Compute the Fourier Series for the following function:

$$f(x) = \begin{cases} 0 & \text{-}\pi \le \text{x} < 0 \\ sin(x) & 0 \le \text{x} \le \pi \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$$

$$a_n = \frac{1}{\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos(nx)dx = 0, \quad n \ge 0.$$

$$b_n = \frac{1}{\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin(nx)dx$$

$$a_0 = \frac{1}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} 0dx + \frac{1}{2\pi}\int_{0}^{\pi} sin(x)dx = \frac{1}{2\pi}\int_{0}^{\pi} sin(x)dx$$

$$a_0 = -\frac{1}{2\pi} * cos(x)\Big|_0^\pi = -\frac{1}{2\pi} * [-1-1] = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} 0\cos(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} sin(x)\cos(nx)dx$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} sin(x)\cos(nx)dx$$

$$2 * sin(x) * cos(nx) = sin((n+1)x) - sin((n-1)x)$$

$$a_n = \frac{1}{2\pi}\int_{0}^{\pi} sin((n+1)x) - sin((n-1)x)dx$$

$$a_n = \frac{1}{2\pi}\int_{0}^{\pi} sin((n+1)x)dx - \frac{1}{2\pi}\int_{0}^{\pi} sin((n-1)x)dx$$

$$a_n = \frac{1}{2\pi}\frac{cos((n+1)x)}{n+1}\Big|_0^\pi - \frac{1}{2\pi}\frac{cos((n-1)x)}{n-1}\Big|_0^\pi$$

$$a_n = \frac{1}{2\pi}[(-1)^{n+1}-1][\frac{1}{n+1} - \frac{1}{n-1}] = \frac{(-1)^{n+1}-1}{\pi(n^2-1)}$$

$$a_n = \begin{cases} 0 & \text{odd} \\ \frac{-2}{\pi(4m^2-1)} & \text{even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} 0\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} sin(x)\sin(nx)dx$$

$$b_n = \frac{1}{\pi}\int_{0}^{\pi} sin(x) * sin(nx)dx$$

$$2 * sin(x) * sin(nx) = cos((n-1)x) - cos((n+1)x)$$

$$b_n = \frac{1}{2\pi}\int_{0}^{\pi} cos((n-1)x) - cos((n+1)x)dx$$

$$b_n = \frac{1}{2\pi}\int_{0}^{\pi} cos((n-1)x)dx - \frac{1}{2\pi}\int_{0}^{\pi} cos((n+1)x)dx$$

$$b_n = \frac{1}{2\pi}\frac{sin((n-1)x)}{n-1}\Big|_0^\pi - \frac{1}{2\pi}\frac{sin((n+1)x)}{n+1}\Big|_0^\pi$$

$$b_n = \frac{sin(n*\pi)}{2\pi*(1-n)} + \frac{sin(n*\pi)}{2\pi*(n+1)}$$

$$b_n = -\frac{sin(n*\pi)}{\pi*(n^2-1)}$$

all b_n is zero except when n=1

$$\lim_{n\to 1} -\frac{sin(n*\pi)}{\pi*(n^2-1)} = 1/2$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$f(x) = \frac{1}{\pi} + \frac{1}{2}sin(x) + \sum_{m=1}^{\infty}\frac{-2}{\pi(4m^2-1)}cos(2mx)$$

$$f(x) = \frac{1}{\pi} + \frac{\sin(x)}{2} - \frac{2\cos(2x)}{3\pi} - \frac{2\cos(4x)}{15\pi} - \frac{2\cos(6x)}{35\pi} - \frac{2\cos(8x)}{63\pi}$$

$$-\frac{2\cos(10x)}{99\pi} - \frac{2\cos(12x)}{143\pi} - \frac{2\cos(14x)}{195\pi} - \frac{2\cos(16x)}{255\pi} - \frac{2\cos(18x)}{323\pi}$$

$$-\frac{2\cos(20x)}{399\pi} - ...$$