Linear Time-Invariant Systems

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1. Let

x[n] = δ[n] + 2δ[n- 1] - δ[n- 3]
h[n] = 2δ[n + 1] + 2δ[n- 1].

Compute and plot each of the following convolutions:

(a) Y_1[n] = x[n] * h[n]

Equation for convolution for discrete and continuous cases

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

Convolution Summation or Superposition summation for Y_1[n] = x[n] * h[n]

$\sum_{k=-\infty}^{\infty} x[n-k] * h[k]$

h[k] is only defined at k = 1 and k = -1 because of equation h[n] = 2δ[n + 1] + 2δ[n- 1].

Y_1[n] = x[n+1] h[-1] + x[n-1] h[1]
Y_1[n] = x[n+1] h[-1] + x[n-1] h[1]
Y_1[n] = 2 * x[n+1] + 2 *x[n-1]
Y_1[n] = 2 (x[n+1] +x[n-1])

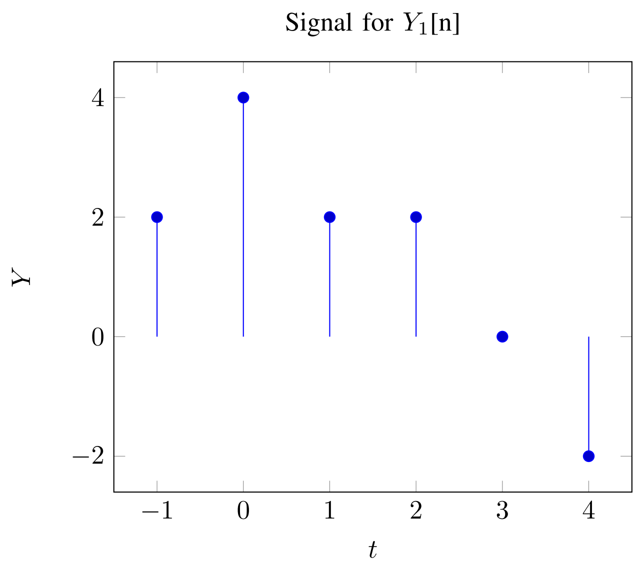
Find [n+1] and x[n-1]
Since x[n] = δ[n] + 2δ[n- 1] - δ[n- 3]

x[n+1] = δ[n+1] + 2δ[n] - δ[n- 2]
x[n-1] = δ[n-1] + 2δ[n-2] - δ[n-4]

Plug in x[n+1], x[n-1] to Y_1[n]

Y_1[n] = 2 (δ[n+1] + 2δ[n] - δ[n- 2] + δ[n-1] + 2δ[n-2] - δ[n-4])

Y_1[n] = 2δ[n+1] +4δ[n] +2δ[n-1] +2δ[n-2] - 2δ[n-4]



2. Consider the signal

$$h[n] = \frac{1}{2}^{n-1} u[n+3] - u[n-10]$$

Express A and B in terms of n so that the following equation holds:

$$h[n-k] = \begin{cases} 0.5^{n-k-1} & \text{A} \leq \text{k} \leq \text{B} \\ 0 & \text{elsewhere} \end{cases}$$

First,

$$u[n+3] - u[n-10] = \begin{cases} 0 & \text{for n<-3} \\ 1 & \text{-3} \leq \text{n} \leq \text{9} \\ 0 & \text{for n>10} \end{cases}$$

For h[k]

$$h[k] = \begin{cases} 0.5^{k-1} & \text{-3} \leq \text{k} \leq \text{9} \\ 0 & \text{elsewhere} \end{cases}$$

For h[-k]

$$h[-k] = \begin{cases} 0.5^{-k-1} & \text{-9} \leq \text{k} \leq \text{3} \\ 0 & \text{elsewhere} \end{cases}$$

For h[n-k]

$$h[n-k] = \begin{cases} 0.5^{n-k-1} & \text{n-9} \leq \text{k} \leq \text{n+3} \\ 0 & \text{elsewhere} \end{cases}$$

Therefore,
A = n-9
B = n+3

3. Consider an input x[n] and a unit impulse response h[n] given by

$$x[n] = 0.5^{n-2}u[n-2]$$

$$h[n] = u[n+2]$$

Determine and plot the output y[n] = x[n] * h[n].

Convolution equation
$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k] * h[k]$

$\sum_{k=-\infty}^{\infty} 0.5^{n-2-k}u[n-2-k] * u[k+2]$

It is defined from when k =2 because of equation h[k] = u[k+2]

y[n]=$\sum_{k=-2}^{\infty} 0.5^{n-2-k}u[n-2-k]$

y[n]= $\sum_{k=-2}^{n-2} 0.5^{n-2-k}$

y[n]= $\sum_{k=0}^{n} 0.5^{n}$

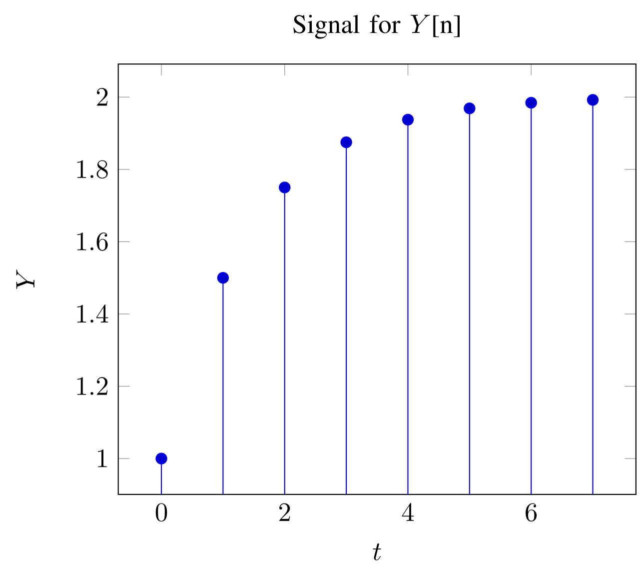
Equation for geometric progression series

$\dfrac{a(1-r^n)}{1-r}$

Therefore,

y(n) = $\dfrac{(1-0.5^n)}{1-0.5}$

y(n) = $2 * (1-0.5^n)$ u[n]



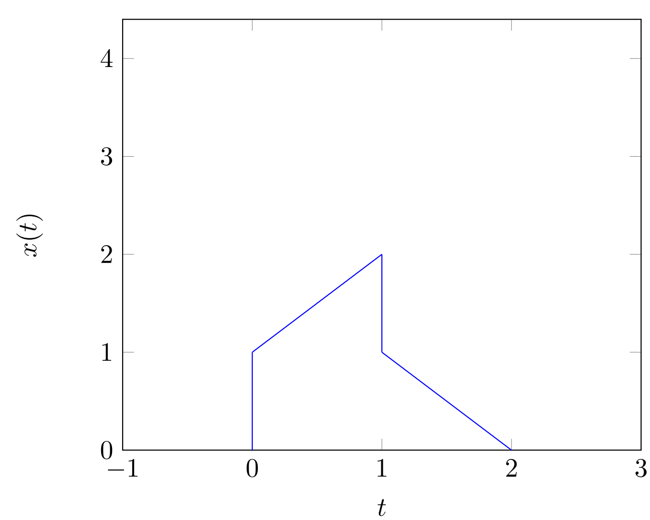
4. Determine and sketch the convolution of the following two signals

$$x(t) = \begin{cases} t+1 & 0 \leq \text{t} \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$h[t] = \delta[t+2] + 2\delta[t+1]$$

For 0 $\leq$ t $\leq$ 1
at t= 0
x(0) = 1
at t = 1
x(1) = 2

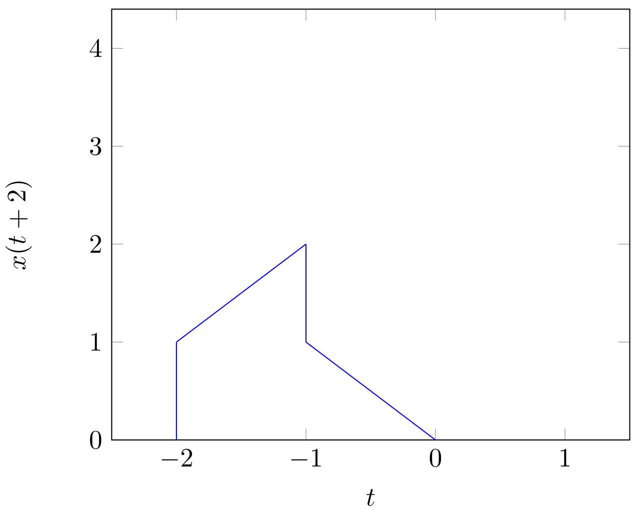
For 1<t≤2
at t = 1
x(1) = 1
at t = 2
x(2) = 0

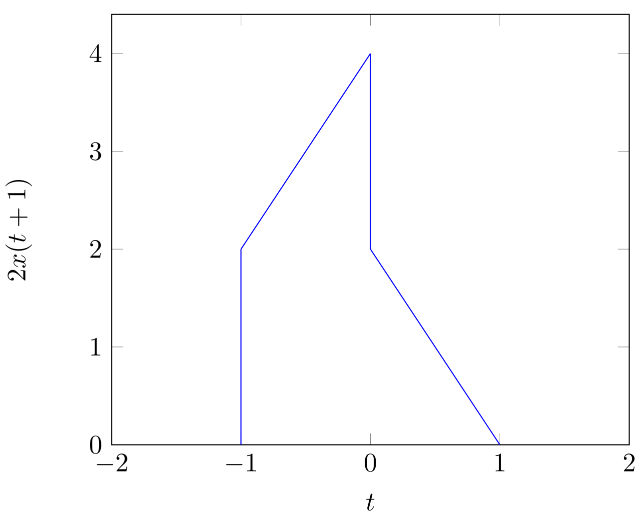


y(t) = x(t) * h(t)
y(t) = x(t)($\delta[t+2] + 2\delta[t+1]$)

Since x(t)*$\delta[t+t_0]$ = $x[t+t_0]$

y(t) = x(t+2) +2x(t+1)



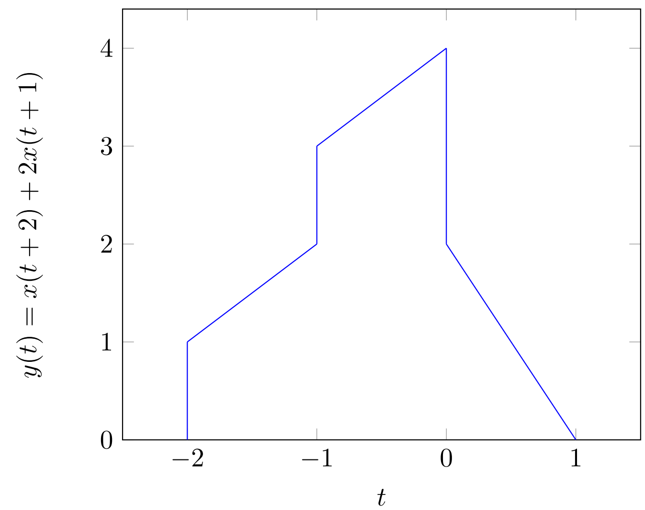


$$x(t) = \begin{cases} t+1 & 0 \leq \text{t} \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$x(t+2) = \begin{cases} t+3 & \text{-2} \leq \text{t} \leq \text{-1} \\ -t & -1 < t \leq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$2x(t+1) = \begin{cases} 2t+4 & \text{-1} \leq \text{t} \leq 0 \\ 2-2t & 0 < t \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$y(t) = x(t+2) + 2x(t+1) = \begin{cases} t+3 & \text{-2} \leq \text{t} \leq \text{-1} \\ t+4 & -1 < t \leq 0 \\ 2-2t & 0 < t \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$



5. Let
x(t) = u(t- 3)- u(t- 5) and h(t) = $e{-}^{3t}$u(t).
(a) Compute y(t) = x(t) * h(t).

y(t) = x(t) * h(t).

y(t) = $\int_{-\infty}^{\infty} x[n-\tau] * h[\tau] d\tau$

y(t) = $\int_{-\infty}^{\infty} [u(t-3-\tau) - u(t-5-\tau)] e^{-3t} u(t) d\tau$

Since u(t) is defined from when t = 0
y(t) = $\int_{0}^{\infty} [u(t-3-\tau) - u(t-5-\tau)] e^{-3t} d\tau$

for 3 $\leq$ t $<$ 5

y(t) = $\int_{0}^{t-3} e^{-3t} d\tau$

y(t) = $-\frac{e^{-3t}}{3}\Big|_{0}^{t-3}$

y(t) = -$\frac{1}{3}(e^{-3(t-3)} - 1)$

y(t) = $\frac{1}{3}(1 - e^{-3(t-3)})$ for 3 $\leq$ t $<$ 5

for t $\leq$ 5

y(t) = $\int_{t-5}^{t-3} e^{-3t} d\tau$

y(t) = $-\frac{e^{-3t}}{3}\Big|_{t-5}^{t-3}$

y(t) = -$\frac{1}{3}(e^{-3(t-3)} - e^{-3(t-5)})$

y(t) = $\frac{1}{3}(e^{-3(t-5)} - e^{-3(t-3)})$ for t $\leq$ 5

$$y(t) = \begin{cases} 0 & -\infty \leq t < 3 \\ 1/3(1-e^{-3(t-3)}) & 3 \leq t < 5 \\ 1/3(e^{-3(t-5)} - e^{-3(t-3)}) & t \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

6. Compute g(t) = (dx(t)/dt) * h(t).

$$x(t) = u(t-3) - u(t-5)$$

$$g(t) = \frac{dx(t)}{dt} * h(t)$$

$$g(t) = [\frac{d}{dt} * (u(t-3) - u(t-5))] * e^{-3t}u(t)$$

$$Since \frac{d}{dt}u(t) = \delta(t)$$

$$g(t) = [\delta(t-3) - \delta(t-5)] * e^{-3t}u(t)$$

$$g(t) = x(t) * h(t). = \int_{-\infty}^{\infty} x[t-\tau] * h[\tau]d\tau$$

$$g(t) = \int_{-\infty}^{\infty} \delta(t-3-\tau) - \delta(t-5-\tau)d\tau * e^{-3\tau}u(\tau)$$

$$g(t) = \int_{-\infty}^{\infty} \delta(t-3-\tau)d\tau * e^{-3\tau}u(\tau)$$

$$-\int_{-\infty}^{\infty} \delta(t-5-\tau)d\tau * e^{-3\tau}u(\tau)$$

Since

$$\int_{-\infty}^{\infty} x(t) * \delta[t-t_0]dt = x(t_0)$$

$$g(t) = e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5)$$

Q.7: 2.13 Consider a discrete-time system S_1 with impulse response

$$h[n] = \frac{1}{5}^n u[n]$$

(a) Find the integer A such that h[n] - Ah[n-1] = δ[n].
Since

$$h[n] = \frac{1}{5}^n u[n]$$

$$h[n-1] = \frac{1}{5}^{n-1} u[n-1]$$

Plug in h[n-1] = $\frac{1}{5}^{n-1}$u[n-1] to h[n] - Ah[n-1] = δ[n]

$$\frac{1}{5}^n u[n] - A\frac{1}{5}^{n-1}[u[n-1]] = \delta[n]$$

consider when n=1

$$\frac{1}{5}^1 u[1] - A\frac{1}{5}^0 u[0] = \delta[0]$$

$$\frac{1}{5} * 1 - A\frac{1}{5}^0 * 1 = 0$$

$$\frac{1}{5} - A = 0$$

$$\frac{1}{5} = A$$

$$A = \frac{1}{5}$$

(b) Using the result from part (a), determine the impulse response g[n] of an LTI system S_2 which is the inverse system of S_1.

$$h[n] - \frac{1}{5}h[n-1] = \delta[n]$$

Definition of inverse system

$$h[n] * g[n] = \delta[n]$$

Convolution of any signal is impulse is the signal itself

$$h[n][\delta[n] - \frac{1}{5}\delta[n-1]] = \delta[n]$$

Thus,

$$g[n] = \delta[n] - \frac{1}{5}\delta[n-1]$$

8. Which of the following impulse responses correspond(s) to stable LTI systems?
(a) h_1(t) = $e^{-(1-2j)t}$u(t)

A system is said to be stable if the impulse response of the system is absolutely integrable.

$$\int_{-\infty}^{\infty} |h_1| dt < \infty$$

$$\int_{-\infty}^{\infty} |e^{-(1-2j)t}u(t)| dt < \infty$$

$$\int_{-\infty}^{\infty} |e^{-t+2jt}u(t)| dt$$

Since the magnitude of the sum of a set of numbers is no larger than the sum of the magnitudes of the numbers, the equation is

$$|y(t)| = \left| \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \right| \leq \int_{-\infty}^{+\infty} |h(\tau)||x(t-\tau)| d\tau$$

$$\int_{-\infty}^{\infty} |e^{-t+2jt}u(t)| dt \leq \int_{-\infty}^{\infty} |e^{-t}| * |e^{2jt}| * |u(t)| dt$$

$$e^{2jt} = cos(2t) + jsin(2t)$$

$$|e^{2jt}| = \sqrt{cos^2(2t) + jsin^2(2t)}$$

= 1

$$\int_{-\infty}^{\infty} |e^{-t}| * 1 * |u(t)| dt = \int_{0}^{\infty} |e^{-t}| dt$$

$$\int_{0}^{\infty} e^{-t} dt = -e^{-t}\Big|_{0}^{\infty}$$

$$-(e^{-\infty} - e^{-0}) = -(0-1)$$

$$\int_{-\infty}^{\infty} |e^{-t+2jt}u(t)| dt \leq 1$$

Since the system is absolutely integrable, system is stable.

9. Consider a causal LTI system whose input x[n] and output y[n] are related by the difference equation

$$y[n] = \frac{1}{4}y[n-1] + x[n]$$

Determine y[n] if

$$x[n] = \delta[n-1]$$

$$h[n] = \begin{cases} 1 & \text{n=1} \\ 0 & \text{elsewhere} \end{cases}$$

Since the System is casual
y[n] = 0 for n<0

$$y[n] = \frac{1}{4}y[n-1] + \delta[n-1]$$

$$y[0] = \frac{1}{4}y[0-1] + \delta[0-1] = \frac{1}{4}y[-1] + \delta[-1]$$

Since

$$\delta[-1] = 0, y[-1] = 0$$

$$y[0] = \frac{1}{4}0 + 0 = 0$$

$$y[1] = \frac{1}{4}y[1-1] + \delta[1-1] = \frac{1}{4}y[0] + \delta[0]$$

Since

$$\delta[0] = 1, y[0] = 1$$

$$y[1] = \frac{1}{4}0 + 1 = 1$$

$$y[2] = \frac{1}{4}y[2-1] + \delta[2-1] = \frac{1}{4}y[1] + \delta[1]$$

Since

$$\delta[1] = 0, y[1] = 1$$

$$y[2] = \frac{1}{4}1 + 0 = \frac{1}{4}$$

$$y[3] = \frac{1}{4}y[3-1] + \delta[3-1] = \frac{1}{4}y[2] + \delta[2]$$

Since

$$\delta[2] = 0, y[2] = \frac{1}{4}$$

$$y[3] = \frac{1}{4} * \frac{1}{4} + 0 = \frac{1}{4}^2$$

$$y[4] = \frac{1}{4}y[4-1] + \delta[4-1] = \frac{1}{4}y[3] + \delta[3]$$

Since

$$\delta[3] = 0, y[3] = \frac{1}{4}^2$$

$$y[4] = \frac{1}{4} * \frac{1}{4}^2 + 0 = \frac{1}{4}^3$$

y[n] exist from when n =1 thus, equation is

$$y[n] = \frac{1}{4}^{n-1} u[n-1]$$

Thus, when x[n] = δ[n - 1], the output of casual LTI system is $\frac{1}{4}^{n-1}u[n-1]$

10. Evaluate the following integrals:

(a)

$$\int_{-\infty}^{\infty} u_0(t) * cos(t)dt$$

$$u_0(t) = \frac{du(t)}{dt} = \delta[t]$$

$$\int_{-\infty}^{\infty} u_0(t) * cos(t)dt = \int_{-\infty}^{\infty} \delta[t] * cos(t)dt$$

Integration only exists when t=0, thus $\int_{-\infty}^{\infty} u_0(t) * cos(t)dt$ exist only when t = 0

$$\int_{-\infty}^{\infty} \delta[t] * cos(t)dt = \delta[0] * cos(0) = 1 * 1 = 1$$