THE CONTINUOUS-TIME FOURIER TRANSFORM

Jiwook Kim

Q.1:(4.5) Use the Fourier transform synthesis equation (4.8) to determine the inverse Fourier transform of $X(j\omega)$ = $|X(j\omega)|e^{j\angle X(J\omega)}$, where

$$|X(j\omega)| = 2\{u(w+3) - u(w-3)\},$$

$$\angle X(J\omega) = -\frac{3}{2}\omega + \pi$$

Use your answer to determine the values of t for which x(t) = 0.

The inverse Fourier transform of X$(j\omega)$ is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|e^{j\angle X(J\omega)}e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\{u(w+3) - u(w-3)\}e^{j(-\frac{3}{2}\omega+\pi)}e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \{u(w+3) - u(w-3)\}e^{-j\frac{3}{2}\omega+j\pi}e^{j\omega t}d\omega$$

$$x(t) = \frac{e^{j\pi}}{\pi}\int_{-3}^{3} e^{-j\frac{3}{2}\omega}e^{j\omega t}d\omega$$

$$e^{j\pi} = -1$$

$$x(t) = \frac{-1}{\pi}\int_{-3}^{3} e^{j\omega(t-\frac{3}{2})}d\omega = \frac{-1}{\pi}[\frac{e^{j\omega(t-\frac{3}{2})}}{j(t-\frac{3}{2})}]\Big|_{-3}^{3}$$

$$x(t) = \frac{-1}{\pi}[\frac{e^{3j(t-\frac{3}{2})} - e^{-3j(t-\frac{3}{2})}}{j(t-\frac{3}{2})}]$$

$$x(t) = \frac{-1}{\pi j(t-\frac{3}{2})}[e^{3j(t-\frac{3}{2})} - e^{-3j(t-\frac{3}{2})}]$$

$$x(t) = \frac{-1}{\pi j(t-\frac{3}{2})}2jsin[3(t-\frac{3}{2})]$$

$$x(t) = -\frac{2sin[3(t-\frac{3}{2})]}{\pi(t-\frac{3}{2})}$$

t when x(t) = 0.

$$-\frac{2sin[3(t-\frac{3}{2})]}{\pi(t-\frac{3}{2})} = 0$$

$$sin[3(t-\frac{3}{2})] = 0$$

$$sin(k\pi) = 0 \text{ where k is integer}$$

$$3(t-\frac{3}{2}) = k\pi$$

$$(t-\frac{3}{2}) = \frac{k\pi}{3}$$

$$t = \frac{k\pi}{3} + \frac{3}{2} \text{ where k is integer}$$

Thus the values of t for which x(t) is zero are $\frac{k\pi}{3} + \frac{3}{2}$

$$x(t) = -\frac{2sin[3(t-\frac{3}{2})]}{\pi(t-\frac{3}{2})}$$

Q.2: 4.6 Given that x(t) has the Fourier transform $X(j\omega)$, express the Fourier transforms of the signals listed below in terms of $X(j\omega)$. You may find useful the Fourier transform properties listed in Table 4.1.

(a) x_1(t) = x(1-t) + x(-1-t)

$$x(-t) <-- \text{Fourier Transform} --> X(-j\omega)$$

$$x(-t+1) <-- \text{Fourier Transform} --> e^{j\omega}X(-j\omega)$$

$$x(-t-1) <-- \text{Fourier Transform} --> e^{-j\omega}X(-j\omega)$$

$$F\{X_1(t)\} = F\{x(1-t) + x(-1-t)\}$$

$$X_1(j\omega) = e^{j\omega}X(-j\omega) + e^{-j\omega}X(-j\omega)$$

$$X_1(j\omega) = [e^{j\omega} + e^{-j\omega}]X(-j\omega)$$

$$Since[e^{j\omega} + e^{-j\omega}] = 2 * cos(\omega)$$

$$X_1(j\omega) = [2 * cos(\omega)]X(-j\omega)$$

Q.3: 4.6(b)

x_2**(t) = x(3t-6)**
x_2(t) = x(3(t-2))

Time scale property

$$x(at) <-- F.T ---> \frac{1}{a}X(\frac{j\omega}{a})$$

Time shifting property

$$x(t-t_0) <-- F.T --> e^{-j\omega}X(j\omega)$$

Calculate the Fourier transform. $X_2(t)$

$$F\{x_2(t)\} = F\{x(3(t-2))\}$$

$$X_2(J\omega) = \frac{1}{3}e^{-j2\omega}X(\frac{j\omega}{3})$$

Q.4: 4.6(c)

x_3**(t) =** $\frac{d^2}{dt^2}$**x(t-1)**

$$\frac{dx(t)}{dt} <-- F.T --> j\omega X(j\omega)$$

$$x(t-t_0) <-- F.T --> e^{-j\omega}X(j\omega)$$

$$\frac{d^2x(t)}{dt^2} <-- F.T --> j\omega[j\omega X(j\omega)]$$

$$\frac{d^2x(t)}{dt^2} <-- F.T --> -\omega^2 X(j\omega)$$

$$x_3(t) = \frac{d^2}{dt^2}x(t-1)$$

$$X_3(j\omega) = e^{-j\omega}[-\omega^2 * X(j\omega)]$$

$$X_3(j\omega) = -\omega^2 e^{-j\omega} * X(j\omega)$$

Q.5: 4.7(a).For each of the following Fourier transforms, use Fourier transform properties (Table 4.1) to determine whether the corresponding time-domain signal is (i) real, imaginary, or neither and (ii) even, odd, or neither. Do this without evaluating the inverse of any of the given transforms.

(a) $X1(j\omega) = u(\omega) - u(\omega - 2)$

The conjugation property allows us to show that if x(t) is real, then X($j\omega$) has conjugate symmetry; that is

$$X_1(j\omega) = X_1^*(-j\omega) \quad \text{[x(t) is real]}$$

Consider the signal $X_1(j\omega)$

$$X_1(j\omega) = u(\omega) - u(\omega - 2)$$

$$X_1^*(-j\omega) = u(-\omega) - u(-\omega - 2)$$

Thus, $X_1(j\omega) \neq X_1^*(-j\omega)$

$X_1(j\omega)$ is not conjugate symmetric, hence the corresponding Signal $x_1(t)$ is not real.

For the signal $x_1(t)$ to be even

$$X_1(-j\omega) = X_1(j\omega)$$

Finding $X_1(-j\omega)$ and knowing $X_1(j\omega) = u(\omega) - u(\omega - 2)$

$$X_1(-j\omega) = u(-\omega) - u(-\omega - 2) \neq X_1(j\omega).$$

Thus x_1(t) is not even

For the signal $x_1(t)$ to be odd

$$X_1(-j\omega) = -X_1(j\omega)$$

Finding $X_1(-j\omega)$ and knowing $X_1(j\omega) = u(\omega) - u(\omega - 2)$

$$X_1(-j\omega) = u(-\omega) - u(-\omega - 2) \neq -X_1(j\omega).$$

Thus x_1(t) is not odd

Since, $X_1(-j\omega) \neq -X_1(j\omega)$. Thus, the signal $x_1(t)$ is not odd.

Hence, the signal $x_1(t)$ is not real, and not even nor odd.

Q.6:4.8(a). Consider the signal

Use the differentiation and integration properties in Table 4.1 and the Fourier transform pair for the rectangular pulse in Table 4.2 to find a closed-form expression for $X(j\omega)$.

$$X(\omega) = \begin{cases} 0 & t < -\frac{1}{2} \\ t + \frac{1}{2} & -\frac{1}{2} < t < \frac{1}{2} \\ 1 & t > \frac{1}{2} \end{cases}$$

$$\frac{dx(\omega)}{dt} = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 & -\frac{1}{2} < t < \frac{1}{2} \\ 0 & t > \frac{1}{2} \end{cases}$$

Fourier transform of the rectangle pulse.

$$Y(j\omega) = 2 * \frac{sin(\frac{\omega}{2})}{\omega}$$

The expression of the signal x(t) is

$$x(t) = \int_{-\infty}^{t} y(t)dt$$

Integration property of Fourier Transform.

$$X(j\omega) = \frac{1}{j\omega}Y(j\omega) + \pi Y(0)\delta(\omega)$$

$$Y(j\omega) = 2 * \frac{sin(\frac{\omega}{2})}{\omega}$$

$$X(j\omega) = \frac{1}{j\omega} * 2 * \frac{sin(\frac{\omega}{2})}{\omega} + \pi * 2 * \frac{sin(\frac{0}{2})}{0}\delta(\omega)$$

$$\text{Since, } \lim_{x \to 0} \frac{sin(x)}{x} = 1$$

$$X(j\omega) = \frac{1}{j\omega} * 2 * \frac{sin(\frac{\omega}{2})}{\omega} + \pi * 2 * 1 * \delta(\omega)$$

$$X(j\omega) = 2 * \frac{sin(\frac{\omega}{2})}{j\omega^2} + 2\pi\delta(\omega)$$

Q.7: 4.10(a). Use Tables 4.1 and 4.2 to help determine the Fourier transform of the following signal:

$$x(t) = t(\frac{sin(t)}{t\pi})^2$$

$$\frac{sin(t)}{\pi t} --F.T--> \text{rectangular function } Y(j\omega)$$

Using Multiplication property

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega$$

$$[\frac{sin(t)}{\pi t}]^2 <--F.T-->$$

$$\frac{1}{2\pi}[\text{rectangular function } Y(j\omega) * \text{rectangular function } Y(j\omega)]$$

$$Y_1(j\omega) = \begin{cases} \frac{\omega}{2\pi} & -2 < \omega < 0 \\ \frac{-\omega}{2\pi} & 0 < \omega < 2 \end{cases}$$

Differentiation in frequency for the Fourier transform

$$\mathcal{F}[\frac{d}{dt}x(t)] = j\omega \; X(j\omega)$$

$$t(\frac{sin(t)}{t\pi})^2 <--F.T--> X(j\omega) = j\frac{d}{d\omega}Y_1(j\omega)$$

$$X(j\omega) = \begin{cases} \frac{j}{2\pi} & -2 < \omega < 0 \\ \frac{-j}{2\pi} & 0 < \omega < 2 \\ 0 & \text{otherwise} \end{cases}$$

Q.8:4.11 Given the relationships

$$y(t) = x(t) * h(t)$$

and

$$g(t) = x(3t) * h(3t)$$

and given that x(t) has Fourier transform X(jw) and h(t) has Fourier transform H(jw), use Fourier transform properties to show that g(t) has the form

$$g(t) = Ay(Bt)$$

Determine the values of A and B.

$$x(at) <--F.T--> \frac{1}{|a|}X(\frac{j\omega}{a})$$

$$x(3t) <--F.T--> \frac{1}{3}X(\frac{j\omega}{3})$$

$$h(3t) <--F.T--> \frac{1}{3}h(\frac{j\omega}{3})$$

$$G(j\omega) = [\frac{1}{3}X(\frac{j\omega}{3})][\frac{1}{3}H(\frac{j\omega}{3})]$$

$$G(j\omega) = \frac{1}{3}[\frac{1}{3}Y(\frac{j\omega}{3})] \quad \text{Since } y(t) = x(t) * h(t)$$

$$g(t) = \frac{1}{3}y(3t) = Ay(Bt)$$

$$A = \frac{1}{3} \qquad B = 3$$

Q.9:4.13(a)
Let x(t) be a signal whose Fourier transform is

$$X(j\omega) = \delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5)$$

(a) Is x(t) periodic?

$$\delta(\omega) <--F.T--> \frac{1}{\pi}$$

$$\delta(\omega - \pi) <--F.T--> \frac{1}{\pi}e^{j\pi t}$$

$$\delta(\omega - 5) <--F.T--> \frac{1}{\pi}e^{j5t}$$

$$x(t) = \frac{1}{2\pi} + \frac{1}{2\pi}e^{j\pi t} + \frac{1}{2\pi}e^{j5t}$$

if $\frac{T_1}{T_2}$ is rational then the signal x(t) is periodic

if $\frac{T_1}{T_2}$ is irrational then the signal x(t) is aperiodic

Period of $e^{j\pi t}$ = $\frac{2\pi}{\pi}$ = 2
Period of e^{j5t} = $\frac{2\pi}{5}$

Ratio $\frac{T_1}{T_2} = \frac{2}{\frac{2\pi}{5}} = \frac{5}{\pi}$

π is irrational number, $\frac{T_1}{T_2}$ is irrational.

Thus, x(t) is not periodic.

Q.10: (4.18) Find the impulse response of a system with the frequency response

$$h(j\omega) = \frac{(sin^2(3\omega))cos(\omega)}{\omega^2}$$

$$h(j\omega) = \frac{(sin^2(3\omega))}{\omega^2}[\frac{e^{j\omega} + e^{-j\omega}}{2}]$$

$$h(j\omega) = \frac{1}{2}e^{j\omega}[\frac{sin^2(3\omega)}{\omega^2}] + \frac{1}{2}e^{-j\omega}[\frac{sin^2(3\omega)}{\omega^2}]$$

$$h(j\omega) = \frac{1}{2}e^{j\omega}[X(j\omega)] + \frac{1}{2}e^{-j\omega}[X(j\omega)]$$

$$X(j\omega) = \frac{sin^2(3\omega)}{\omega^2}$$

$$X(j\omega) = \frac{1}{4}\frac{2sin(3\omega)}{\omega} * \frac{2sin(3\omega)}{\omega}$$

$$X(j\omega) = \frac{1}{4}X_1(j\omega)X_1(j\omega)$$

$$H(j\omega) = \frac{1}{2}e^{j\omega}[X(j\omega)] + \frac{1}{2}e^{-j\omega}[X(j\omega)]$$

$$h(t) = \frac{1}{2}x(t+1) + \frac{1}{2}x(t-1)$$

$$X_1(j\omega) = \frac{2sin(3\omega)}{\omega}$$

$$x_1(j\omega) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$$

$$x_1(j\omega) = \begin{cases} 1 & |t| < 3 \\ 0 & |t| > 3 \end{cases}$$

$$X(j\omega) = \frac{1}{4}[x_1(t) * x_1(t)]$$

$$\frac{1}{4}\int_{-\infty}^{\infty} x_1(\tau)x_1(t-\tau)d\tau$$

when t<-6

$$\frac{1}{4}\int_{-\infty}^{\infty} x_1(\tau)x_1(t-\tau)d\tau = 0$$

$$\frac{1}{4}\int_{-\infty}^{\infty} 0 * x_1(t-\tau)d\tau = 0$$

when $-6 \leq t \leq 0$, then $x_1(\tau)$*$x_1(t-\tau)$ equal to 1 in region -3 $\leq\tau\leq$t+3.

$$-3 \leq \tau \leq t+3$$

$$\frac{1}{4}\int_{-\infty}^{\infty} x_1(\tau)x_1(t-\tau)d\tau$$

$$\frac{1}{4}\int_{-3}^{t+3} 1d\tau$$

$$\frac{1}{4}t|_{-3}^{t+3}$$

$$\frac{1}{4}[t+3-(-3)]$$

$$x(t) = \frac{1}{4}t + \frac{3}{2}$$

when $0 \leq t \leq 6$, then $x_1(\tau)$*$x_1(t-\tau)$ equal to 1 in region t-3 $\leq\tau\leq$3.

$$t-3 \leq \tau \leq 3$$

$$\frac{1}{4}\int_{-\infty}^{\infty} x_1(\tau)x_1(t-\tau)d\tau$$

$$\frac{1}{4}\int_{t-3}^{3} 1d\tau$$

$$\frac{1}{4}t|_{t-3}^{3}$$

$$\frac{1}{4}[3-(t-3)]$$

$$x(t) = -\frac{1}{4}t + \frac{3}{2}$$

when t>6

$$\frac{1}{4}\int_{-\infty}^{\infty} x_1(\tau)x_1(t-\tau)d\tau = 0$$

$$\frac{1}{4}\int_{6}^{\infty} 0 * x_1(t-\tau)d\tau = 0$$

$$x_1(t) = \begin{cases} 0 & t < -6 \\ \frac{t}{4} + \frac{3}{2} & -6 \leq t \leq 0 \\ -\frac{t}{4} + \frac{3}{2} & 0 \leq t \leq 6 \\ 0 & t > 6 \end{cases}$$

$$\frac{1}{2}x_1(t+1) = \begin{cases} 0 & t < -7 \\ \frac{t}{8} + \frac{7}{8} & -7 \leq t \leq -1 \\ -\frac{t}{8} + \frac{5}{8} & -1 \leq t \leq 5 \\ 0 & t > 5 \end{cases}$$

$$\frac{1}{2}x_1(t-1) = \begin{cases} 0 & t < -5 \\ \frac{t}{8} + \frac{5}{8} & -5 \leq t \leq 1 \\ -\frac{t}{8} + \frac{7}{8} & 1 \leq t \leq 7 \\ 0 & t > 7 \end{cases}$$

$$h(t) = \frac{1}{2}x_1(t+1) + \frac{1}{2}x_1(t-1)$$

$$h(t) = \begin{cases} 0 & t < -7 \\ \frac{t}{8} + \frac{7}{8} & -7 \leq t \leq -5 \\ \frac{t}{4} + \frac{3}{2} & -5 \leq t \leq -1 \\ \frac{5}{4} & -1 \leq t \leq 1 \\ -\frac{t}{4} + \frac{3}{2} & 1 \leq t \leq 5 \\ -\frac{t}{8} + \frac{7}{8} & 5 \leq t \leq 7 \\ 0 & t > 7 \end{cases}$$